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DETERMINATION OF PERIOD-3 CYCLE VIA SYLVESTER MATRIX IN THE CHAOTIC REGION OF A ONE DIMENSIONAL MAP

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ABSTRACT

In science, for a long time, it has been assumed that regularity therefore predictability has been the centre of approaches to explain the behaviours of systems. Whereas in real life, it is a well known fact that systems exhibit unexpected behaviours which lead to irregular and unpredictable outcomes. This approach, named as non-linear dynamics, produces much closer representation of real happenings. The chaos theory which is one of methods of non-linear dynamics, has recently attracted many scientist from all different fields. In this paper we analyse a situation in which the sequence { $f^n(x)$} is non-periodic and might be called "chaotic". Here we have considered a one parameter map (Verhulst population model), obtained the parameter value $\boldsymbol{\lambda}$for which period-3 cycle is created in a *Tangent bifurcation*, using Sarkovskii's Theorem, Sylvester's Matrix and Resultant. We also calculated the parameter range $\boldsymbol{\lambda}_0<\boldsymbol{\lambda}<\boldsymbol{\lambda}_1$ for which the map possesses stable period-3 orbit.

KEYWORDS: Period 3 Cycle, Tangent Bifurcation, Sarkovskii's Theorem, Sylvester's Matrix, Resultant, Bifurcation Diagram, Chaos

1 INTRODUCTION

When determining whether a function is chaotic under iteration, it is necessary to prove that the set of all periodic points is a dense subset of the space on which the function acts. If the function is simple enough, it may be possible to explicitly construct a periodic orbit that comes within an arbitrary distance of any given point. However, often this is not possible, and so a more general approach is needed.

Sarkovskii's Theorem provides a means of proving the existence of infinitely many cycles, each with different period, provided that we can find a cycle of length $k \neq 2^n, n \epsilon \mathbb{Z}^+$[1,4]

In 1975, the article: "Period three implies chaos", was published in the American Mathematical Monthly by Li and Yorke [5]. (Period three means that there is a point x_0 such that $f^3(x_0) = f\left(f\left(f(x_0)\right)\right) = x_0, f^k(x_0) \neq x_0$ for $k = 1,2;$in other words, the image of x_0 comes back to x_0 after three iterations.)In that article, Li and Yorke announced that a new theorem for continuous functions of a single variable was discovered. The theorem states that if a continuous function has period three, it must have period n for every positive integer n. Soon afterwards, it was found that Li and Yorke's theorem is only a special case of the remarkable theorem published a decade earlier by Soviet mathematician A.N. Sarkovskii, in a Ukrainian journal. Sarkovskii reordered the natural numbers and proved that if $l \lhd m$ (which means l is "less than" m in Sarkovskii's ordering) and if a function has period l then it must have period

m. The number 3 is the 'smallest' in Sarkovskii's ordering. So, obviously, period 3 implies all the other periods, and Li-Yorke's theorem was not a new one. However, it was in Li-Yorke's article that the new concept of chaos was first introduced into mathematics. People were surprised that iterations of even a very simple continuous function of a single variable can display extremely complicated chaotic behavior [**14].**

Intermediate Value Theorem

If f is continuous on [a,b] and N is any number between f(a) and f(b), then there exists at least one x_0 between a and b such that $f(x_0) = N$.

Proposition 1.1: Let f be continuous on [a,b]. If the range of f contains [a,b], then equation f(x)=x has at least one solution in [a,b].

The solution is straightforward. Since the range of f contains [a,b], there must be some $x_1, x_2 \in [a,b]$ such that $f(x_1) \leq a \leq x_1, f(x_2) \geq b \geq x_2$.

Let g(x)=f(x)-x. The result follows from applying the intermediate value theorem with N=0.

By the way, if the assumption "the range of f contains [a,b]" is replaced by "the range of f is contained in [a,b]," proposition 1.1 is still valid.

A point x_0 satisfying proposition (1.1) is called a fixed point. A natural generalization of fixed point is periodic point.

Assume$R\{f\} \subset D\{f\}$, the range of f is contained in the domain of f. Denote

$$f^0(x) = x, f^1(x) = f(x), f^2(x) = f\big(f(x)\big), f^3(x) = f\big(f^2(x)\big), \dots, f^n(x) = f\big(f^{n-1}(x)\big).$$

If x_0 satisfies $\begin{cases} f^n(x_0) = x_0 \\ f^k(x_0) \neq x_0 \end{cases} \quad k = 1,2,\dots,n-1,$ (1.1)

thenx_0 is called an n-periodic point with period n. Clearly, a fixed point is a 1-periodic point.

If x_0 is an n-periodic point of f, then $x_0, f(x_0), \dots, f^{n-1}(x_0)$ are distinct and the set $\{x_0, f(x_0), \dots, f^{n-1}(x_0)\}$ is called a periodic orbit of f.

If f has an n-periodic point, we say that f has period n.

The existence of a fixed point of a function is generally clear by the inspection of its graph. But the existence of an n-periodic point is not so easy to see even if n is a small integer. As an example, let us consider the function

$$\psi(x) = \begin{cases} x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} < x \leq 1 \end{cases} \tag{1.2}$$

Whose graph is in Figure 1.a.

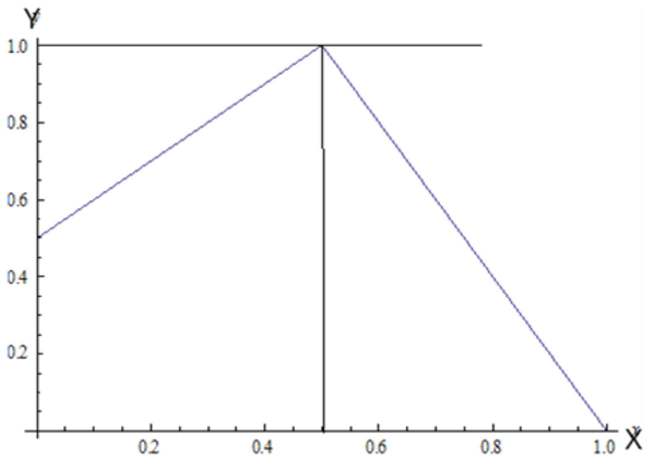

Figure 1a

As it can be seen, $\psi(0) = \frac{1}{2}, \psi^2(0) = \psi\left(\frac{1}{2}\right) = 1, \psi^3(0) = \psi^2\left(\frac{1}{2}\right) = \psi(1) = 0$; that is ψ has a 3-periodic point 0. Does it have a 5-periodic point? A 7-periodic point? It is hard to ascertain this by just looking at the graph. We need to do some deeper analysis.

The following is a generalized version of intermediate value theorem.

Proposition 1.2: Let f be continuous on [a,b], let $l_0, l_1, \dots, l_{n-1}$ be closed subintervals of [a,b]. If

$$f(I_k) \supset I_{k+1,} k = 0,1, \dots, n-2,$$

$$f(I_{n-1}) \supset I_0 \quad (1.3)$$

then, the equation $f^n(x) = x$ (1.4)

has at least one solution $x = x_0 \epsilon I_0$ such that $f^k(x_0) \epsilon I_k$, k=0,1,...,n-1. (1.5)

In the proposition, $f(I_k) \supset I_{k+1}$ means that the range of f on I_k contains I_{k+1}.

We will use the notation $I_i \to I_j$ or $I_j \leftarrow I_i$ (1.6)

If $f(I_i) \supset I_j$ ($f(I_i)$"covers" I_j). The condition (1.3) can be written as

$$I_0 \to I_1 \to I_2 \to \cdots \to I_{n-1} \to I_0$$

Clearly, if n=1, proposition 1.2 is reduced to proposition 1.1.

The proof of proposition 1.2 is based on the following fact:

If $I_1 \to I_2$, then there exists a subinterval $I_1' \subset I_1$ such that $f(I_1') = I_2$.

This is true, since if I_2.=[c,d], there exist x_1 and x_2 in I_1 such that f(x_1)=c and f(x_2)=d.

Let $I_1' = [x_1, x_2]$. Then $I_1' \subset I_1$, and by the intermediate value theorem, $f(I_1') = I_2$.

This fact implies that there exist $I_{n-1}' \subset I_{n-1}$ such that $f(I_{n-1}') = I_0$,$I_{n-2}' \subset I_{n-2}$ such that

$f(I_{n-2}') = I_{n-1}'$,..., and $I_0' \subset I_0$ such that $f(I_0') = I_1'$. In other words, there exist $I_k' \subset I_k$ such that

$f(I_k') = I'_{k+1} \subset I_{k+1}$ for k=0,1, ...,n-2

And $f(I'_{n-1}) = I_0 \supset I_0'$ (1.7)

From (1.7), $f^k(I_0') = I_k'$, for k=0,1,...,n-2, and $f^k(I_0') \supset I_0'$. Thus, by proposition 1.1, equation (1.4) has a solution $x_0 \in I_0' \subset I_0$ such that (1.5) holds.

Note: In geometry, (1.5) means that mapped successively by f, x_0 visits $I_1, I_2, \dots, I_{n-1}$ and finally comes back to where it was.

Proposition 1.2 itself is not a remarkable result. But it is the only calculus which is needed for the proof of famous two results as follows in theorem 1.3 and theorem 1.4.

Period Three and Chaos

The famous Li-Yorke theorem is the following:

Theorem1.3: Let f be continuous on [a,b], its range contained in [a,b]. If f has a 3-periodic point, then f has periodic points of period k for all positive integer values k.

Proof: The proof is an outline due to Devaney[8].Let a, b, c $\in$ R and suppose f(a) = b, f(b) = c, and f(c) = a. We assume that a < b < c. The only other possibility, f(a) = c, is handled similarly. Let I_0 = [a,b] and I_1 = [b, c].From the assumption $f(I_0) \supseteq I_0$ and $f(I_1) \supseteq [a,c]$,which suffices that f has a fixed point in (b,c).

Similarly it can be shown that f^2 has fixed point in the interval (b,c), or f has a periodic point of period 2 in the interval (b,c). Again, let f has a periodic point of period n≥ 2.As already it is assumed that f has a periodic point of period 3,so it is necessary that f has a prime period >3.Indeed it can be obtained by constructing a nested sequence of intervalsA_n.

Sarkovskii's Theorem

Before formally stating Sarkovskii's Theorem, it is necessary to define Sarkovskii's Ordering. This ordering of the natural numbers begins with all odd numbers, written in increasing order. These are followed by 2 times the odds, 2^2times the odds, 2^3times the odds, and so on. The powers of 2 come last, in decreasing order. This ordering can be written as follows:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots$$

$$\rhd 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd 2^3\cdot 7 \cdots\cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Theorem 1.4: (Sarkovskii,1964). Let $f: I \to I$ be continuous and let f have an l-periodic point. If $l \rhd m$, then f has an m-periodic point, too.

Proof: The outline of the proof can be given due to Devaney[8] by starting with an assumption that f has a periodic point x of period n with n odd and n >1 and f has no periodic points of odd period less thann.Let $x_1, x_2, \dots, x_n$be the points on the orbit of x, enumerated from left to right such that f permutes the x_is. Then it is observed that a sequence of intervals I_n

can be constructed such that $I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow \cdots I_n \rightarrow I_1$.In fact periods larger than n are given by the sequence $I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow \cdots I_n \rightarrow I_1 \rightarrow I_1 \rightarrow I_1$ ....The smaller even periods are given by cycles of the form$I_{n-1} \rightarrow I_{n-2} \rightarrow I_{n-1}$ or $I_{n-1} \rightarrow I_{n-4} \rightarrow I_{n-3} \rightarrow I_{n-2} \rightarrow I_{n-1}$.Hence the theorem for n is odd.For the case of n even it is clear that f must have a periodic point of period two. Then for n=2^m,let k=$2^\wedge l$, l<m. Let g=$f^{k/2}$,by assumption g has a periodic point of period 2^{m-l+1},hence g has a periodic point of period 2 say x*,i.e.g(g(x*))=x*,i.e. $f^k(x^*) = x^*$,which means x* is a point of period 2^l.n = $p\ 2^m$ where p is odd, can bereduced to the previous two.

However it can be shown that period 5 may not imply period 3[8].

2 THE VERHULST POPULATION MODEL

The quadratic expression $x + \lambda x(1-x)$ has a very interesting interpretation and history in biology. It serves as the core of a population dynamics model which in spirit goes back to the Belgian mathematician Pierre Francois Verhulst and his work around 1845.

The most simple population model would assume a constant growth rate, but in that situation we find unlimited growth which is not realistic. In our model we will assume that the population is restricted by a constant environment, but this premise requires a modification of the growth law. Now the growth rate depends on the actual size of the population relative to its maximal size. Verhulst postulated that the growth rate at time n should be proportional to the difference between the population count and the maximal population size, which is a convenient measure for the fraction of the environment that is not yet used up by the population at time n. This assumption leads to the Verhulst population model

$$x_{n+1} = x_n + rx_n(1-x_n) \tag{2.1}$$

Where the actual population at time n is denoted by X_n and x_n measures the relative population count $x_n = \dfrac{X_n}{N}$ and N is the maximal population size which can be supported by the environment.

Growth rate is measured by the quantity $\dfrac{X_{n+1} - X_n}{X_n}$ (2.2)

In other words, the growth rate "r" at time n measures the increase of the population in one time step relative to the size of the population at time n.

2.1 Derivation of the Verhulst Model

This population model assumes that the growth rate depends on the current size of the population. First we normalize the population count by introducing $x = \dfrac{X}{N}$.Thus x ranges between 0 and 1, i.e. we can interpret x=0.06, for example, as the population size being 6% of its maximal size N. Again we index x by n, i.e. write x_n to refer tothe size at time steps n=0,1,2,3… Now growth rate is measured by the quantity already given corresponding to the expression (2.2), $\dfrac{x_{n+1} - x_n}{x_n}$.

Verhulst postulated that the growth rate at time n should be proportional to $1-x_n$ (the fraction of the environment that is not yet used by the population at time n). Assuming that the population is restricted by a constant environment the growth should change according to the following table.

Table 1

Population	Growth Rate
Small	Positive, large
About 1	Small
Less than 1	Positive
Greater than 1	Negative

In other words, $\frac{x_{n+1}-x_n}{x_n} \propto 1-x_n$,

Or, after introducing a suitable constant , $\frac{x_{n+1}-x_n}{x_n}$ =r$(1-x_n)$, solving the equation yields the population model equation (2.1) $x_{n+1}=x_n+rx_n(1-x_n)$.

2.2 Period-3 Cycle and Tangent Bifurcation

We are going to discuss one aspect of the map

$$f_r(x_n)=x_{n+1}=x_n+rx_n(1-x_n) \qquad (2.3)$$

namely the value of "r" at which a period-3 cycle is created in a tangent bifurcation.

Let r_0 be the smallest positive value of "r" for which the equation $f_r^3(x)=x$ has a solution x_0 that is not already a fixed point of f_r.The sequence $x_0, x_1=f_r(x_0), x_2=f_r(x_1)$ is called a period-3 orbit of f_r.Since $f^3(x)$ is an eight-degree polynomial, this equation is not explicitly solvable.But a graph provides sufficient insight .

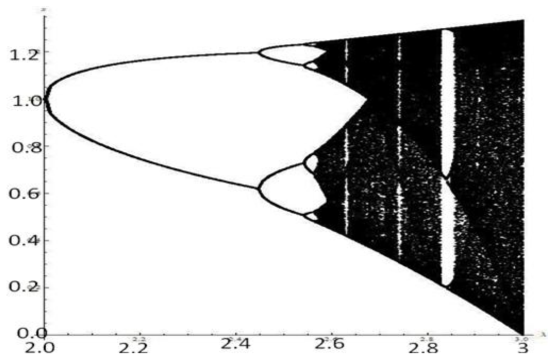

Figure 2a: Bifurcation Diagram for the Parameter "r"for $2 \le r \le 3$

Period 3 orbit at the tangent bifurcation can be obtained from the following two equations:

$$f^3(x)=x,$$

$$\frac{d}{dx}f^{3(x)}=1$$

i.e we have two polynomial equations

$$f^3(x)-x=0,$$

$$\frac{d}{dx} f^{3(x)} - 1 = 0$$

To solve the above two equations we follow the following procedure:

2.3 The Sylvester Matrix and Resultants

Let *f* (*x*) ∈R[*x*] and *g*(*x*) ∈R[*x*] be univariate polynomials with real coefficients. We want to determine whether *f* and *g* have a common zero. We know already one technique for solving the problem: We compute the gcd of *f* and *g*. It comprises exactly the common roots of *f* and *g*. The gcd of *f* and *g* does not only tell us whether *f* and *g* have common roots; it tells us how many common roots there are and it is a compact description of the common roots. In this section, we will see an alternative technique, the resultant calculus. In its basic form, it will only tell us whether *f* and *g* have a common root; it will not tell us how many common roots there are nor will it give a description of the common roots. In this sense, resultants are weaker than greatest common divisors. They are stronger in the sense, that they can give us information about common roots of multivariate polynomials.

The resultant is an algebraic tool used for analysis and derivation of various algorithms associated with the greatest common divisor (gcd) problem. The resultant made a wide impact on many algebraic algorithms and today it has generalizations to more than two polynomials, to matrix and to multivariate polynomials. The technique which is used for computing resultantswas established by J.J. Sylvester in 1840.

Common Zeros of Univariate Polynomials

Assume that $f(x)$and $g(x)$have a common factor *h*. Then $f = \left(\frac{f}{h}\right) h$and $g = \left(\frac{g}{h}\right) h$and hence$f.\frac{g}{h} = \frac{f}{h}.h.\frac{g}{h} = \frac{f}{h}.g$ or $f.\frac{g}{h} - \frac{f}{h}.g \equiv 0$

In other words, we have nonzero polynomials $s = \frac{g}{h}$and $t = -\frac{f}{h}$such that

$$0 \leq deg\ s < deg\ g \text{ and } 0 \leq deg\ t < deg\ f \text{ and } fs + gt \equiv 0 \quad (2.4)$$

We have thus proved one direction of the following Lemma.

LEMMA 2.3.1: Let *f ∈R[x]*and *g ∈R[x]*be univariate polynomials. *f* and *g* have a common zero if and only if there are polynomials *s* and *t* satisfying (2.4).

Proof. First part is already proved .

Conversely

Assume that there are *s* and *t* satisfying (2.4). i.e. $0 \leq deg\ s < deg\ g$ and $0 \leq deg\ t < deg\ f$and $fs + gt \equiv 0.$ Now using linear algebra we can find *s* and *t* as in (13) or check for their existence.

Let $\deg f = n$and $\deg g = m$

And let $g(x) = b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \cdots\cdots\cdots\cdots\cdots + b_1 x^1 + b_0$

$$s(x) = s_{m-1} x^{m-1} + s_{m-2} x^{m-2} + s_{m-3} x^{m-3} + \cdots\cdots\cdots\cdots\cdots\cdots + s_1 x^1 + s_0$$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + a_1 x^1 + a_0$$

$$t(x) = t_{n-1}x^{n-1} + t_{n-2}x^{n-2} + t_{n-3}x^{n-3} + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + t_1 x^1 + t_0$$

Where we restrict the coefficients of *s* and *t* to R

Let $P(x) = f(x)s(x) + g(x)t(x)$

Then

$$P(x) = (a_n s_{m-1} + b_m t_{n-1})x^{m+n-1} + (a_n s_{m-2} + a_{n-1}s_{m-1} + b_m t_{n-2} + b_{m-1}t_{n-1})x^{m+n-2} + (a_n s_{m-3} + a_{n-1}s_{m-2} + a_{n-2}s_{m-1} + b_m t_{n-3} + b_{m-1}t_{n-2} + b_{m-2}t_{n-1})x^{m+n-3} + \cdots\cdots\cdots\cdots + (a_0 s_0 + b_0 t_0)x^0$$

.

We want $P(x) \equiv 0$. This is equivalent to the following $n+m$ linear equations for the $n+m$ coefficients of *s* and *t*

$$a_n s_{m-1} + b_m t_{n-1} = 0$$

$$a_n s_{m-2} + a_{n-1}s_{m-1} + b_m t_{n-2} + b_{m-1}t_{n-1} = 0$$

$$a_0 s_0 + b_0 t_0 = 0$$

The system of linear equations can be written in matrix form as:

$$(s_{m-1}, s_{m-2}, \cdots\cdots\cdots, s_1, s_0, t_{n-1}, t_{n-2}, \cdots\cdots\cdots, t_1, t_0)\ Syl\ (f,g) = 0 \qquad (2.5)$$

Where

$$Syl(f,g) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & . & . & . & a_0 & 0 & . & . & . & . & 0 \\ 0 & a_n & a_{n-1} & a_{n-2} & . & . & . & a_0 & 0 & . & . & . & 0 \\ 0 & 0 & a_n & a_{n-1} & a_{n-2} & . & . & . & a_0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & a_n & . & . & . & . & . & a_1 & a_0 \\ b_m & b_{m-1} & . & . & . & . & . & b_0 & 0 & . & . & . & 0 \\ 0 & b_m & b_{m-1} & . & . & . & . & . & b_0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & b_m & b_{m-1} & . & . & . & . & b_1 & b_0 \end{bmatrix}$$

is the Sylvester matrix of f and g. This is a square matrix with n+m rows and columns.

We know from linear algebra that the system (2.5) has a nontrivial solution it and only if the determinant of the co-efficient matrix (i.e. the Sylvester matrix) is zero.

The determinant of the Sylvester matrix will play an important role in the sequel and hence deserves a name. The resultant *res (f, g)* of *f* and *g* is defined as the determinant of the Sylvester matrix of *f* and *g*, i.e., *res (f, g) = determinant of Syl (f, g)*.

We now have an elegant condition for *f* and *g* having a common zero.

Theorem 2.3.2: Let $f(x), g(x) \epsilon \mathbb{R}[x]$. Then f and g have a common zero if and only if $res(f,g) = 0$

Proof: From lemma 2.3.1, we get *res(f,g) = determinant of Syl(f ,g) and* the system (2.5) has a nontrivial solution it and only if the determinant of the co-efficient matrix (i.e. the Sylvester matrix) is zero $\Rightarrow res(f,g) = 0$

Theorem 2.3.3: Let $f(x) = 0$ be a polynomial equation. Then a solution x_1 , of $f(x) = 0$ is a solution of $f'(x) = 0$ if and only if at $x = x_1$, the curve of the function $y = f(x)$ touches the X-axis.

Proof: Let x_1 be a solution of $f(x) = 0$. Then $f'(x_1) = 0$. This implies at $x = x_1$, the tangent to the curve $y = f(x)$ is parallel to the X-axis. (2.6)

But since $f(x) = 0$ [As x_1 is a solution of $f(x) = 0$].Therefore, at $x = x_1$, the curve of the function $y = f(x)$ is either intersects the X-axis or touches the X-axis (2.7)

From (2.6) and (2.7), we have at $x = x_1$, the curve of the function $y = f(x)$ touches the X-axis.

Conversely, let at $x = x_1$, the curve of the function $y = f(x)$ touches the X-axis.Then, the tangent to the curve $y = f(x)$ is nothing but the X-axis.This implies, the slope of the tangent to the curve $y = f(x)$ is zero. This further implies that $f'(x_1) = 0$. Hence the result.

2.4 Calculating the Range of "r" Values for which f_r Possesses Stable Period -3 Orbits

Any point of a period-3 orbit will satisfy $f^3(x) = x$ and for some critical value of "r" period-3 cycles are created in a tangent bifurcation. At a tangent bifurcation, $f^3(x)$ has slope 1 at each of the three points of the period-3 cycle at $r = r_0$ [3]. Since $f^3(x)$ is tangent to $y = x$ at r$= r_0$, it follows that at r_0 , $\frac{d}{dx} f^3(x_1) = +1$. The value of r_0 can be determined from $G_r(x) = f^3(x) - x$ and $\frac{d}{dx} G_r(x) = \frac{d}{dx} f^3(x_1) - 1$.

For special choice of r$= r_0$, $G_r(x) = f^3(x) - x$ is tangent to the x-axis. Hence by the above theorems $G_r(x)$ and $\frac{d}{dx} G_r(x)$ have common roots when r$= r_0$, which implies that the resultant of $G_r(x)$ and $\frac{d}{dx} G_r(x)$ must vanish. Therefore $R\left(G_r(x), \frac{d}{dx} G_r(x)\right) = 0$.

If we denote

$$f_r^3(x) = -r^7x^8 + (4r^6 + 4r^7)x^7 + (-8r^5 - 14r^6 - 6r^7)x^6 + (10r^4 + 24r^5 + 18r^6 + 4r^7)x^5 + (-9r^3 - 25r^4 - 25r^5 - 10r^6 - r^7)x^4 + (6r^2 + 18r^3 + 20r^4 + 10r^5 + 2r^6)x^3 + (-3r - 9r^2 - 10r^3 - 5r^4 - r^5)x^2 + (1 + 3r + 3r^2 + r^3)x$$

$$G_r(x) = f_r^3(x) - x$$

$$= -r^7x^8 + (4r^6 + 4r^7)x^7 + (-8r^5 - 14r^6 - 6r^7)x^6 + (10r^4 + 24r^5 + 18r^6 + 4r^7)x^5 + (-9r^3 - 25r^4 - 25r^5 - 10r^6 - r^7)x^4 + (6r^2 + 18r^3 + 20r^4 + 10r^5 + 2r^6)x^3 + (-3r - 9r^2 - 10r^3 - 5r^4 - r^5)x^2 + (3r + 3r^2 + r^3)x$$

Then for the special choice $r = r_0$, G_r is tangent to the x-axis. In other words, when $r = r_0$, the polynomials G_r and $\frac{d}{dx} G_r$ have a common root, which implies that the resultant of G_r and $\frac{d}{dx} G_r$ must vanish.

Here,

$$\frac{d}{dx} G_r = -8r^7x^7 + (28r^6 + 28r^7)x^6 + (-48r^5 - 84r^6 - 36r^7)x^5 + (50r^4 + 120r^5 + 90r^6 + 20r^7)x^4 + (-36r^3 - 100r^4 - 100r^5 - 40r^6 - 4r^7)x^3 + (18r^2 + 54r^3 + 60r^4 + 30r^5 + 6r^6)x^2 + (-6r - 18r^2 - 20r^3 - 10r^4 - 2r^5)x + (3r + 3r^2 + r^3)$$

Using Sylvester's theorem and a computer algebra system, we are going to find out $R(G_r, \frac{d}{dx} G_r)$. **Sylvester's Matrix(S) is shown below**

$$\begin{bmatrix}
-r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 \\
-8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 3r+3r^2+r^3
\end{bmatrix}$$

$R\left(G_r, \frac{d}{dx} G_r\right) =$

$$Det(s) = -r^{15}(-3359232r^{36} + 5738688r^{38} - 5569560r^{40} + 3606849r^{42} - 1718172r^{44} + 617274r^{46} - 169848r^{48} + 35331r^{50} - 5408r^{52} + 570r^{54} - 36r^{56} + r^{58}) = -r^{51}(r^2 - 8)^3(r^2 - 3r + 3)^4(r^2 + 3r + 3)^4$$

The only real, positive root of this polynomial that does not correspond to a fixed point of f_r occurs at

$$r_0 = 2\sqrt{2} \approx 2.82843$$

For "r" slightly greater than r_0, one of the corresponding period-3 orbits is stable, or attracting .This means that if x_0 is a value in this orbit ,then $|(f_r{}^3)'(x_0)| < 1$,and for x sufficiently close to x_0, $f^{3n}(x) \rightarrow x_0$ as $n \rightarrow \infty$.

Eventually, as "r" increases beyond a certain value, this corresponding orbit becomes unstable or repelling. When this change or bifurcation occurs, the slope of $f_r{}^{(3)}$ at each point in this orbit equals -1. Now the corresponding value of "r" can be determined using

$R\left(f_r{}^{(3)}(x) - x, \frac{d}{dx} f_r{}^{(3)} + 1\right).$

Where

$$\frac{d}{dx} f_r{}^3 = -8r^7x^7 + (28r^6 + 28r^7)x^6 + (-48r^5 - 84r^6 - 36r^7)x^5 + (50r^4 + 120r^5 + 90r^6 + 20r^7)x^4 + (-36r^3 - 100r^4 - 100r^5 - 40r^6 - 4r^7)x^3 + (18r^2 + 54r^3 + 60r^4 + 30r^5 + 6r^6)x^2 + (-6r - 18r^2 - 20r^3 - 10r^4 - 2r^5)x + (1 + 3r + 3r^2 + r^3)$$

$$f_r{}^3(x) - x = -r^7x^8 + (4r^6 + 4r^7)x^7 + (-8r^5 - 14r^6 - 6r^7)x^6 + (10r^4 + 24r^5 + 18r^6 + 4r^7)x^5 + (-9r^3 - 25r^4 - 25r^5 - 10r^6 - r^7)x^4 + (6r^2 + 18r^3 + 20r^4 + 10r^5 + 2r^6)x^3 + (-3r - 9r^2 - 10r^3 - 5r^4 - r^5)x^2 + (3r + 3r^2 + r^3)x$$

Sylvester's Matrix is shown below

$$
\left[\begin{array}{ccccccccccccccc}
-r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -r^7 & 4r^6+4r^7 & -8r^5-14r^6-6r^7 & 10r^4+24r^5+18r^6+4r^7 & -9r^3-25r^4-25r^5-10r^6-r^7 & 6r^2+18r^3+20r^4+10r^5+2r^6 & -3r-9r^2-10r^3-5r^4-r^5 & 3r+3r^2+r^3 & 0 \\
-8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8r^7 & 28r^6+28r^7 & -48r^5-84r^6-36r^7 & 50r^4+120r^5+90r^6+20r^7 & -36r^3-100r^4-100r^5-40r^6-4r^7 & 18r^2+54r^3+60r^4+30r^5+6r^6 & -6r-18r^2-20r^3-10r^4-2r^5 & 2+3r+3r^2+r^3
\end{array}\right]
$$

R $\left(f_r^{(3)}(x) - x, \frac{d}{dx} f_r^{(3)} + 1\right)$

$= \mathrm{Det}(S) = -(r-2)r^{49}(r+2)(r^2-r+1)(1+r+r^2)(r^6-11r^4+37r^2-108)^3$

The real, positive root of this polynomial that does not correspond to a fixed point or to a period-2 orbit of f_r is the single real, positive root r_1 of

$$r^6 - 11r^4 + 37r^2 - 108 = 0 \qquad (2.8)$$

Only two real roots are there

$r_1 = 2.8415$ and $r_2 = -2.8415$

Hence f_r possesses stable period -3 orbits when $r_0 < r < r_1$.

$[r_0 = 2\sqrt{2} \approx 2.82843 \; ; \; r_1 = 2.8415]$

Figure 2.a plots $f^3(x)$ for $r =$ 2.8515. Intersections between the graph and the diagonal line y = x correspond to solutions of $f^3(x) = x$. There are eight solutions,of which two are not genuine period-3,they are period-1 points for which $f(x^*) = x^*$.

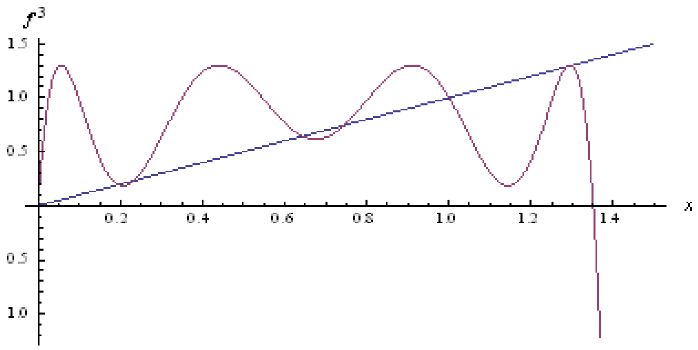

Figure 2b: Graph of $f^3(x)$ for r= 2.8515

Figure 2.b Shows that when r=2.8, the six marked intersections have vanished.Somewhere between r=2.8 and r=2.8415,the graph of $f^3(x)$ must have become tangent to the diagonal.At this critical value of r, the period-3 cycles are created in a tangent bifurcation.

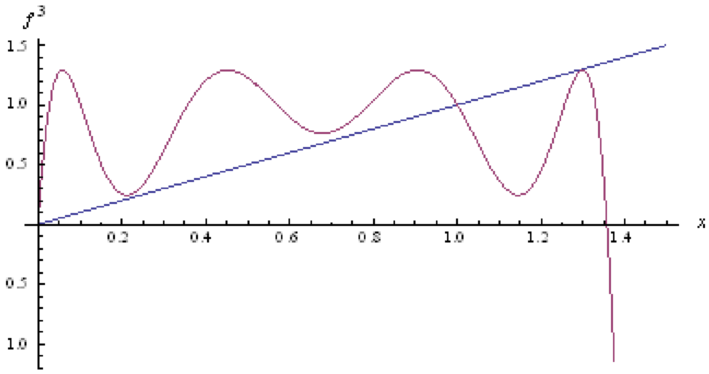

Figure 2c: Graph of $f^3(x)$ for r=2.8 (The Period 3 Cycle has Disappeared)

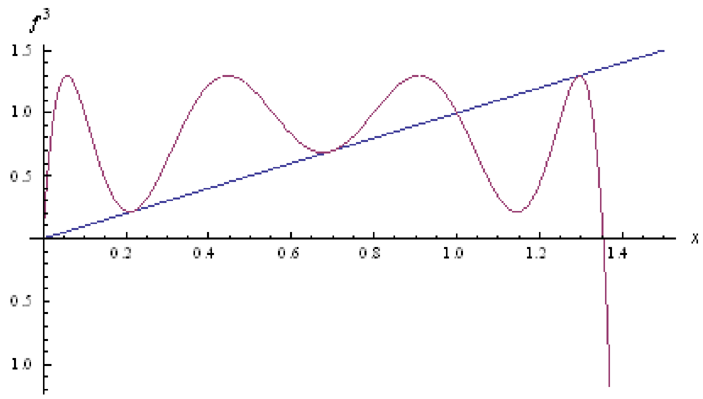

Figure 2d: Graph of $f^3(x)$ for r= $2\sqrt{2}$ =2.82843 (The Birth of Period 3 Orbit)

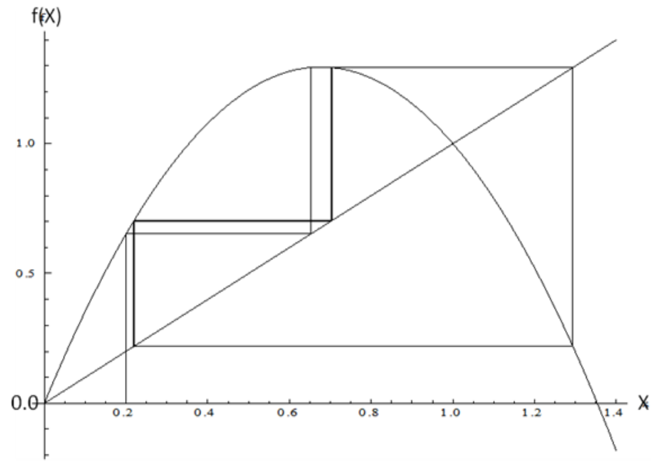

Figure 2e: Cobweb Diagram of the Map f(x)=x+rx(1-x) for r=2.82843 (Along the x-Axis , x Values and along the y-Axis the Value of f(x))

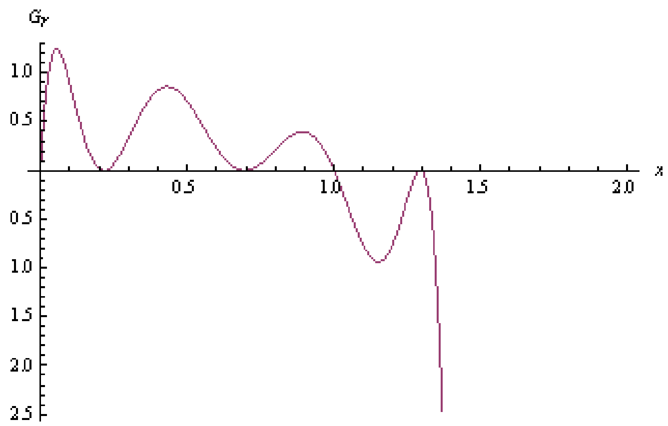

Figure 2f: Graph of $G_r(x)$

Comment on the Orbit Diagram of the Map

As “r” continuously increases from 0 to $2\sqrt{2}$ (onset of the 3-cycle), periodic orbits of periods from bottom up in the Sarkovskii ordering gradually come into existence. Every time a new cycle is born, it is attracting, while the previous one becomes repelling. In the process, the periodic orbits are never destroyed. Rather they just become repelling. Thus the set of periodic orbits steadily gets larger as “r” increases. When $r > 2\sqrt{2}$, this set contains periodic orbits of any prime period.

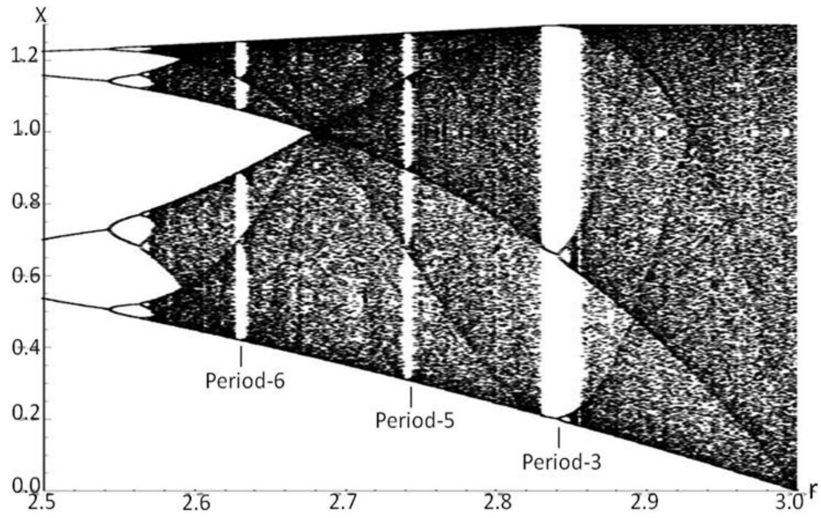

Figure 2g: Bifurcation Diagram for $2.5 < r < 3$

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